ゼータ関数（Zeta function）

$$\zeta(s)=\sum_1^{\infty}\frac{1}{n^s}=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots$$

に対して，交代級数（alternating series）

$$\phi(s)=\sum_1^{\infty}\frac{(-1)^{s-1}}{n^s}=1-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\cdots$$

を考え，変形すると，

$$\begin{aligned}\phi(s)&=\left(1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots\right)-2\left(\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\cdots\right)\\&=\left(1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots\right)-\frac{2}{2^s}\left(1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots\right)\\&=\left(1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots\right)\left(1-\frac{2}{2^s}\right)=\zeta(s)(1-2^{1-s})\end{aligned}$$

すなわち，$\phi(s)=(1-2^{1-s})\cdot\zeta(s)$

の関係を導くことができる。一方，マクローリン展開から，

$$f_0(x)=\frac{1}{1-x}=1+x+x^2+x^3+x^4+\cdots$$

であるから，これを微分すると

$$\frac{d(f_0(x))}{dx}=\frac{1}{(1-x)^2}=1+2x+3x^2+4x^3\cdots$$

だから，

$$f_1(x)=x\frac{d(f_0(x))}{dx}$$

とおくと，

$$f_1(x)=x\frac{d(f_0(x))}{dx}=\frac{x}{(1-x)^2}=x+2x^2+3x^3+4x^4\cdots$$

同様に,

$$f_2(x) = x\frac{d(f_1(x))}{dx} = \frac{x(1+x)}{(1-x)^3} = x + 2^2x^2 + 3^2x^3 + 4^2x^4 \cdots$$

$$f_3(x) = x\frac{d(f_2(x))}{dx} = \frac{x(1+4x+x^2)}{(1-x)^4}$$

$$= x + 2^3x^3 + 3^3x^3 + 4^3x^4 \cdots$$

$$f_4(x) = x\frac{d(f_3(x))}{dx} = \frac{x(1+11x+11x^2+x^3)}{(1-x)^5}$$

$$= x + 2^4x^2 + 3^4x^3 + 4^4x^4 \cdots$$

$$f_5(x) = x\frac{d(f_4(x))}{dx} = \frac{x(1+26x+66x^2+26x^3+x^4)}{(1-x)^6}$$

$$= x + 2^5x^2 + 3^5x^3 + 4^5x^4 \cdots$$

$$f_6(x) = x\frac{d(f_5(x))}{dx} = \frac{x(1+57x+302x^2+302x^3+57x^4+x^5)}{(1-x)^7}$$

$$= x + 2^6x^2 + 3^6x^3 + 4^6x^4 \cdots$$

$$f_7(x) = x\frac{d(f_6(x))}{dx} = \frac{x(1+120x+1191x^2+2416x^3+1191x^4+120x^5+x^6)}{(1-x)^8}$$

$$= x + 2^7x^2 + 3^7x^3 + 4^7x^4 \cdots$$

これらの関数に－１を代入した値と交代級数との関係をみると,

$$f_0(-1) = \phi(0) = -\frac{1}{2}, \quad f_1(-1) = -\phi(-1) = -\frac{1}{4},$$

$$f_2(-1) = -\phi(-2) = 0, \quad f_3(-1) = -\phi(-3) = \frac{1}{8}$$

$$f_4(-1) = \phi(-4) = 0, \quad f_5(-1) = -\phi(-5) = -\frac{1}{4}$$

$$f_6(-1) = -\phi(-2) = 0, \quad f_7(-1) = -\phi(-3) = \frac{17}{16}$$

一方，交代級数とゼータ関数との関係から

$$\phi(0)=(1-2^{1-0})\zeta(0)=(1-2)\zeta(0)=-\zeta(0),$$

$$\phi(-1)=(1-2^{1-(-1)})\zeta(-1)=(1-4)\zeta(-1)=-3\zeta(-1),$$

$$\phi(-2)=(1-2^{1-(-2)})\zeta(-2)=(1-8)\zeta(-2)=-7\zeta(-2),$$

$$\phi(-3)=(1-2^{1-(-3)})\zeta(-3)=(1-16)\zeta(-3)=-15\zeta(-3)$$

$$\phi(-4)=(1-2^{1-(-4)})\zeta(-4)=(1-32)\zeta(-4)=-31\zeta(-4)$$

$$\phi(-5)=(1-2^{1-(-5)})\zeta(-5)=(1-64)\zeta(-5)=-63\zeta(-5)$$

$$\phi(-6)=(1-2^{1-(-6)})\zeta(-6)=(1-128)\zeta(-6)=-127\zeta(-6)$$

$$\phi(-7)=(1-2^{1-(-7)})\zeta(-7)=(1-256)\zeta(-7)=-255\zeta(-7)$$

したがって

$$\zeta(0)=-\phi(0)=-\frac{1}{2},$$

$$\zeta(-1)=-\frac{1}{3}\phi(-1)=-\frac{1}{3}\times\frac{1}{4}=-\frac{1}{12},$$

$$\zeta(-2)=-\frac{1}{7}\phi(-2)=-\frac{1}{7}\times 0=0,$$

$$\zeta(-3)=-\frac{1}{15}\phi(-3)=-\frac{1}{15}\times\left(-\frac{1}{8}\right)=\frac{1}{120}$$

$$\zeta(-4)=-\frac{1}{31}\phi(-4)=-\frac{1}{31}\times 0=0$$

$$\zeta(-5)=-\frac{1}{63}\phi(-5)=-\frac{1}{63}\times\left(\frac{1}{4}\right)=-\frac{1}{252}$$

$$\zeta(-6)=-\frac{1}{127}\phi(-6)=-\frac{1}{15}\times 0=0$$

$$\zeta(-7)=-\frac{1}{255}\phi(-7)=-\frac{1}{15}\times\left(-\frac{17}{16}\right)=\frac{17}{4080}$$

すなわち，

$$\zeta(0)=\sum_{n=1}^{\infty}\frac{1}{n^0}=1+1+1+1+\cdots \qquad =-\frac{1}{2},$$

$$\zeta(-1)=\sum_{n=1}^{\infty}\frac{1}{n^{-1}}=1+2+3+4+\cdots \qquad =-\frac{1}{12},$$

$$\zeta(-2)=\sum_{n=1}^{\infty}\frac{1}{n^{-2}}=1^2+2^2+3^2+4^2+\cdots \quad =0,$$

$$\zeta(-3)=\sum_{n=1}^{\infty}\frac{1}{n^{-3}}=1^3+2^3+3^3+4^3+\cdots \quad =\frac{1}{120},$$

$$\zeta(-4)=\sum_{n=1}^{\infty}\frac{1}{n^{-4}}=1^4+2^4+3^4+4^4+\cdots \quad =0$$

$$\zeta(-5)=\sum_{n=1}^{\infty}\frac{1}{n^{-5}}=1^5+2^5+3^5+4^5+\cdots \quad =-\frac{1}{252},$$

$$\zeta(-6)=\sum_{n=1}^{\infty}\frac{1}{n^{-6}}=1^6+2^6+3^6+4^6+\cdots \quad =0,$$

$$\zeta(-7)=\sum_{n=1}^{\infty}\frac{1}{n^{-7}}=1^7+2^7+3^7+4^7+\cdots \quad =\frac{17}{4080},$$

一方，$f_0(x)=\dfrac{1}{1-x}=1+x+x^2+x^3+x^4+\cdots$

を積分すると，

$$\int f_0(x)dx=-\log(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+$$

となるので，以下のようにおくことができる。

$$\zeta(1)=\sum_{n=1}^{\infty}\frac{1}{n^1}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=-\log(0)=\infty$$

【もうひとつの繰り込み】

$$g(x)=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\cdots=$$

$$=1+x(1+x+x^2+x^3+\cdots)=1+xg(x)$$

$$\therefore(1-x)g(x)=1 \quad \to g(x)=\frac{1}{1-x}$$

x＝－１，２を代入して，

$$g(-1)=\frac{1}{1-(-1)}=\frac{1}{2}=1-1+1-1+\cdots$$

$$g(2)=\frac{1}{1-2}=-1=1+2+4+8+\cdots$$